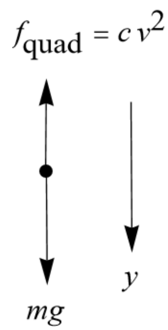
Problem 2.23

Find the terminal speeds in air of **(a)** a steel ball bearing of diameter 3 mm, **(b)** a 16-pound steel shot, and **(c)** a 200-pound parachutist in free fall in the fetal position. In all three cases, you can safely assume the drag force is purely quadratic. The density of steel is about 8 g/cm^3 and you can treat the parachutist as a sphere of density 1 g/cm^3.

Solution

Draw a free-body diagram for a mass falling down in a medium with quadratic air resistance. Let the positive y-direction point downward.



Apply Newton's second law in the y-direction.

$$\sum F_y = ma_y$$

Let $v_y = v$ to simplify the notation.

$$mg - cv^2 = m\frac{dv}{dt}$$

The terminal speed occurs when the velocity reaches equilibrium.

$$mg - cv_{\text{ter}}^2 = m(0)$$

Solve for the terminal velocity.

$$v_{\text{ter}} = \sqrt{\frac{mg}{c}}$$

For a spherical projectile in air at STP, the coefficient of air resistance is $c = \gamma D^2$, where $\gamma = 0.25\ \text{N} \cdot \text{s}^2/\text{m}^4$.

$$v_{\text{ter}} = \frac{2}{D}\sqrt{mg}$$

This is the formula to start with for parts (b) and (c) since the weight is given, but for part (a), the formula needs to be written in terms of diameter.

$$v_{\text{ter}} = \frac{2}{D}\sqrt{\varrho V g}$$

Use the formula for the volume of a sphere.

$$v_{\text{ter}} = \frac{2}{D}\sqrt{\varrho\left[\frac{4}{3}\pi\left(\frac{D}{2}\right)^3\right]g}$$

$$= \frac{2}{D}\sqrt{\frac{\pi\varrho g D^3}{6}}$$

$$= \sqrt{\frac{2\pi\varrho g D}{3}}$$

Note that all variables need to be in SI units.

Part (a)

For a steel ball bearing of diameter 3 mm, which has density 8 g/cm^3,

$$v_{\text{ter}} = \sqrt{\frac{2\pi\varrho g D}{3}}$$

$$= \sqrt{\frac{2\pi}{3}\left[8\ \frac{\text{g}}{\text{cm}^3} \times \frac{1\ \text{kg}}{1000\ \text{g}} \times \left(\frac{100\ \text{cm}}{1\ \text{m}}\right)^3\right]\left(9.81\ \frac{\text{m}}{\text{s}^2}\right)\left(3\ \text{mm} \times \frac{1\ \text{m}}{1000\ \text{mm}}\right)}$$

$$\approx 22.2\ \frac{\text{m}}{\text{s}}.$$

Part (b)

A weight of steel, 16 pounds, is given here rather than a diameter.

$$mg = \varrho V g$$

$$= \varrho\left[\frac{4}{3}\pi\left(\frac{D}{2}\right)^3\right]g$$

$$= \frac{\pi\varrho D^3}{6}g$$

Solve for D.

$$D = \sqrt[3]{\frac{6m}{\pi\varrho}}$$

Therefore,

$$\begin{aligned}
v_{\text{ter}} &= \frac{2}{D}\sqrt{mg} \\
&= \frac{2}{\sqrt[3]{\frac{6m}{\pi\varrho}}}\sqrt{mg} \\
&= \sqrt[3]{\frac{4\pi\varrho}{3}}\sqrt[6]{m}\sqrt{g} \\
&= \sqrt[3]{\frac{4\pi}{3}\left[8\ \frac{\text{g}}{\text{cm}^3}\times\frac{1\ \text{kg}}{1000\ \text{g}}\times\left(\frac{100\ \text{cm}}{1\ \text{m}}\right)^3\right]}\sqrt[6]{16\ \text{lb}\times\frac{1\ \text{kg}}{2.204622476\ \text{lb}}}\sqrt{9.81\ \frac{\text{m}}{\text{s}^2}} \\
&\approx 141\ \frac{\text{m}}{\text{s}}.
\end{aligned}$$

Part (c)

Another weight is given (200 pounds) with density 1 g/cm^3, so use the same formula from part (b) to determine the terminal velocity.

$$\begin{aligned}
v_{\text{ter}} &= \sqrt[3]{\frac{4\pi\varrho}{3}}\sqrt[6]{m}\sqrt{g} \\
&= \sqrt[3]{\frac{4\pi}{3}\left[1\ \frac{\text{g}}{\text{cm}^3}\times\frac{1\ \text{kg}}{1000\ \text{g}}\times\left(\frac{100\ \text{cm}}{1\ \text{m}}\right)^3\right]}\sqrt[6]{200\ \text{lb}\times\frac{1\ \text{kg}}{2.204622476\ \text{lb}}}\sqrt{9.81\ \frac{\text{m}}{\text{s}^2}} \\
&\approx 107\ \frac{\text{m}}{\text{s}}
\end{aligned}$$